

Fractional Brownian Motion Limit for a Model of Turbulent Transport

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Abstract Passive scalar motion in a family of random Gaussian velocity fields with long-range correlations is shown to converge to persistent fractional Brownian motions in long times.

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Abbreviated title fractional Brownian motion limit.

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1 Introduction

We consider the motion of a passive scalar advected by a random velocity field $\mathbf{V}(t, \mathbf{x}) = (V_1(t, \mathbf{x}), \dots, V_d(t, \mathbf{x}))$. The governing equation is

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{V}(t, \mathbf{x}(t)) \quad (1)$$

where $\mathbf{V}(t, \mathbf{x})$ is a mean-zero, time-stationary, space-homogeneous random incompressible velocity field.

In certain situations, it is believed that the *convergence* of the Taylor-Kubo formula ([14], [8]) given by

$$\int_0^\infty \{\mathbf{E}[V_i(t, \mathbf{0})V_j(0, \mathbf{0})] + \mathbf{E}[V_i(t, \mathbf{0})V_j(0, \mathbf{0})]\} dt \quad (2)$$

is a criterion for convergence of passive scalar motion to Brownian motion in the long time limit. Indeed, it has been shown that the solution of

$$\frac{d\mathbf{x}^\varepsilon(t)}{dt} = \frac{1}{\varepsilon} \mathbf{V}\left(\frac{t}{\varepsilon^2}, \mathbf{x}^\varepsilon(t)\right), \quad \mathbf{x}^\varepsilon(0) = \mathbf{0} \quad (3)$$

converges in law, as $\varepsilon \rightarrow 0$, to the Brownian motion with diffusion coefficients given by the Taylor-Kubo formula when the velocity field is sufficiently *mixing in time* (see [7], [6], [9], [2]). Moreover, the solution of (3) converges to the same Brownian motion for a family of *non-mixing* Gaussian, Markovian flows with power-law spectra as long as the Taylor-Kubo formula converges (see [3]). In this paper, for the same family of power-law spectra, we show that, when the Taylor-Kubo formula *diverges*, the solution of the following equation

$$\frac{d\mathbf{x}_\varepsilon(t)}{dt} = \varepsilon^{1-2\delta} \mathbf{V}\left(\frac{t}{\varepsilon^{2\delta}}, \mathbf{x}_\varepsilon(t)\right), \quad \mathbf{x}^\varepsilon(0) = \mathbf{0}, \quad (4)$$

with some $\delta \neq 1$ depending on the velocity spectrum, converges, as $\varepsilon \rightarrow 0$, to a fractional Brownian motion (FBM), as introduced in [10] (see also [13]).

We define the family of velocity fields with power-law spectra as follows. Let (Ω, \mathcal{V}, P) be a probability space of which each element is a velocity field $\mathbf{V}(t, \mathbf{x})$, $(t, \mathbf{x}) \in R \times R^d$ satisfying the following properties.

H 1) $\mathbf{V}(t, \mathbf{x})$ is time stationary, space-homogeneous and centered, i.e., $\mathbf{E}\{\mathbf{V}\} = \mathbf{0}$, and Gaussian. Here \mathbf{E} stands for the expectation with respect to the probability measure P .

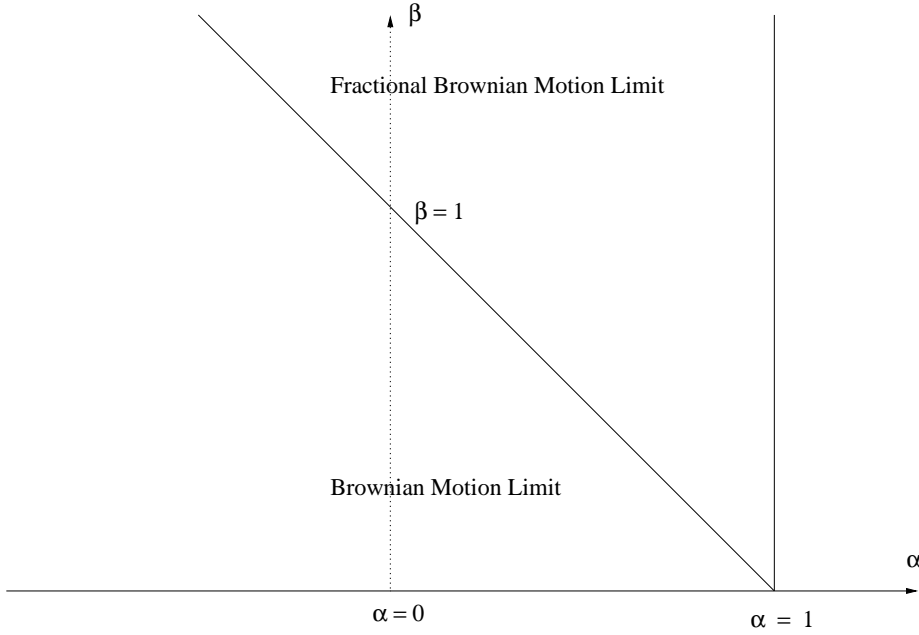
H 2) The two-point correlation tensor $\mathbf{R} = [R_{ij}]$ is given by

$$R_{ij}(t, \mathbf{x}) = \mathbf{E}[V_i(t, \mathbf{x})V_j(0, \mathbf{0})] = \int_{R^d} \cos(\mathbf{k} \cdot \mathbf{x}) e^{-|\mathbf{k}|^{2\beta} t} \hat{\mathbf{R}}_{ij}(\mathbf{k}) d\mathbf{k} \quad (5)$$

with the spatial spectral density

$$\hat{\mathbf{R}}(\mathbf{k}) = \frac{a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}} \left(\mathbf{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2} \right), \quad (6)$$

where $a : [0, +\infty) \rightarrow R_+$ is a compactly supported, continuous, nonnegative function. The factor $\mathbf{I} - \mathbf{k} \otimes \mathbf{k}/|\mathbf{k}|^2$ in (6) is a result of incompressibility.



H 3) $\alpha < 1$, $\beta \geq 0$ and $\alpha + \beta > 1$.

The function $\exp(-|\mathbf{k}|^{2\beta}t)$ in (5) is called the *time correlation function* of the flow \mathbf{V} . For $\beta > 0$, the velocity field lacks the spectral gap and, thus, is not mixing in time. As the time correlation function is exponential, the Gaussian velocity field is Markovian in time.

Because the function a has a compact support we may assume, without loss of generality, that \mathbf{V} is jointly continuous in both (t, \mathbf{x}) and is C^∞ in \mathbf{x} almost surely. For $\alpha < 1$, the spectral density $\hat{\mathbf{R}}(\mathbf{k})$ is integrable in \mathbf{k} and, thus, (5)-(6) defines a random velocity field with a finite second moment. The exponent α is directly related to the decay exponent of \mathbf{R} . Namely $|\mathbf{R}|(0, \mathbf{x}) \sim |\mathbf{x}|^{\alpha-1}$ for $|\mathbf{x}| \gg 1$. As α increases to one, the decay exponent of \mathbf{R} decreases to zero.

Our main result is summarized in the following theorem.

Theorem 1 *Under the assumptions H 1)- H 3), the solution of eq. (4) with the scaling exponent*

$$\delta := \frac{\beta}{\alpha + 2\beta - 1}$$

converges in law, as ε tends to zero, to a fractional Brownian motion $\mathbf{B}_H(t)$ that is to a Gaussian process with stationary increments whose covariance is given by

$$\mathbf{E}[\mathbf{B}_H(t) \otimes \mathbf{B}_H(t)] = \mathbf{D}t^{2H}, \quad (7)$$

with the coefficients \mathbf{D}

$$\mathbf{D} = \int_{\mathbb{R}^d} \frac{e^{-|\mathbf{k}|^{2\beta}} - 1 + |\mathbf{k}|^{2\beta}}{|\mathbf{k}|^{2\alpha+4\beta-1}} \left(\mathbf{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2} \right) \frac{a(0)}{|\mathbf{k}|^{d-1}} d\mathbf{k} \quad (8)$$

and the Hurst exponent H

$$1/2 < H = 1/2 + \frac{\alpha + \beta - 1}{2\beta} < 1. \quad (9)$$

Remark. Molecular diffusion can be added to the equation of motion so that instead of (1) we may consider an Itô stochastic differential equation

$$d\mathbf{x}(t) = \mathbf{V}(t, \mathbf{x}(t))dt + \sqrt{2\kappa}d\mathbf{B}(t)$$

with $\mathbf{B}(t)$, $t \geq 0$ the standard Brownian motion, independent of \mathbf{V} and $\kappa \geq 0$. This however would not influence our results.

2 Multiple stochastic integrals

By the Spectral Theorem (see, e.g., [1]) we assume without loss of any generality that there exist two independent, identically distributed, real vector valued, Gaussian spectral measures $\hat{\mathbf{V}}_l(t, \cdot)$, $l = 0, 1$ such that

$$\mathbf{V}(t, \mathbf{x}) = \int \hat{\mathbf{V}}_0(t, \mathbf{x}, d\mathbf{k}), \quad (10)$$

where

$$\hat{\mathbf{V}}_0(t, \mathbf{x}, d\mathbf{k}) := c_0(\mathbf{k} \cdot \mathbf{x})\hat{\mathbf{V}}_0(t, d\mathbf{k}) + c_1(\mathbf{k} \cdot \mathbf{x})\hat{\mathbf{V}}_1(t, d\mathbf{k})$$

with $c_0(\phi) \equiv \cos(\phi)$, $c_1(\phi) \equiv \sin(\phi)$. Define also

$$\hat{\mathbf{V}}_1(t, \mathbf{x}, d\mathbf{k}) := -c_1(\mathbf{k} \cdot \mathbf{x})\hat{\mathbf{V}}_0(t, d\mathbf{k}) + c_0(\mathbf{k} \cdot \mathbf{x})\hat{\mathbf{V}}_1(t, d\mathbf{k}).$$

We have the relations

$$\partial \hat{\mathbf{V}}_0(t, \mathbf{x}, d\mathbf{k}) / \partial x_j = k_j \hat{\mathbf{V}}_1(t, \mathbf{x}, d\mathbf{k}), \quad (11)$$

$$\partial \hat{\mathbf{V}}_1(t, \mathbf{x}, d\mathbf{k}) / \partial x_j = -k_j \hat{\mathbf{V}}_0(t, \mathbf{x}, d\mathbf{k}). \quad (12)$$

Clearly $\int \hat{\mathbf{V}}_1(t, \mathbf{x}, d\mathbf{k})$ is a random field distributed identically to and independently of \mathbf{V} .

We define the *multiple stochastic integral*

$$\int \cdots \int \psi(\mathbf{k}_1, \dots, \mathbf{k}_N) \widehat{\mathbf{V}}_{l_1}(t_1, \mathbf{x}_1, d\mathbf{k}_1) \otimes \cdots \otimes \widehat{\mathbf{V}}_{l_N}(t_N, \mathbf{x}_N, d\mathbf{k}_N) \quad (13)$$

for any $l_1, \dots, l_N \in \{0, 1\}$ and a suitable family of functions ψ by using the Fubini theorem (see (14) below). For $\psi_1, \dots, \psi_N \in \mathcal{S}(R^d)$, the Schwartz space, and $l_1, \dots, l_N \in \{0, 1\}$ we set

$$\begin{aligned} & \int \cdots \int \psi_1(\mathbf{k}_1) \cdots \psi_N(\mathbf{k}_N) \widehat{\mathbf{V}}_{l_1}(t_1, \mathbf{x}_1, d\mathbf{k}_1) \otimes \cdots \otimes \widehat{\mathbf{V}}_{l_N}(t_N, \mathbf{x}_N, d\mathbf{k}_N) \\ &:= \int \psi_1(\mathbf{k}_1) \widehat{\mathbf{V}}_{l_1}(t_1, \mathbf{x}_1, d\mathbf{k}_1) \otimes \cdots \otimes \int \psi_N(\mathbf{k}_N) \widehat{\mathbf{V}}_{l_N}(t_N, \mathbf{x}_N, d\mathbf{k}_N). \end{aligned} \quad (14)$$

We then extend the definition of multiple integration to the closure \mathcal{H} of the Schwartz space $\mathcal{S}((R^d)^N, R)$ under the norm

$$\|\psi\|^2 := \int \cdots \int \psi(\mathbf{k}_1, \dots, \mathbf{k}_N) \psi(\mathbf{k}'_1, \dots, \mathbf{k}'_N) \quad (15)$$

$$\mathbf{E} \left[\widehat{\mathbf{V}}_{l_1}(t_1, \mathbf{x}_1, d\mathbf{k}_1) \otimes \cdots \otimes \widehat{\mathbf{V}}_{l_N}(t_N, \mathbf{x}_N, d\mathbf{k}_N) \cdot \widehat{\mathbf{V}}_{l_1}(t_1, \mathbf{x}_1, d\mathbf{k}'_1) \otimes \cdots \otimes \widehat{\mathbf{V}}_{l_N}(t_N, \mathbf{x}_N, d\mathbf{k}'_N) \right].$$

The expectation is to be calculated by the formal rule

$$\mathbf{E} \left[\widehat{V}_{l,i}(t, \mathbf{x}, d\mathbf{k}) \widehat{V}_{l',i'}(t', \mathbf{x}', d\mathbf{k}') \right] = e^{-|\mathbf{k}|^{2\beta}|t-t'|} \delta_{l,l'} c_0(\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')) \widehat{R}_{i,i'}(\mathbf{k}) \delta(\mathbf{k} - \mathbf{k}') d\mathbf{k} d\mathbf{k}'.$$

This approach to spectral integration follows [12].

When $\mathbf{i} = (i_1, \dots, i_d), i_1, \dots, i_d \in \{1, 2, \dots, d\}$ is fixed and $\mathbf{l} = (l_1, \dots, l_N), l_1, \dots, l_N \in \{0, 1\}$ we shall denote the corresponding component of the stochastic integral by $\Psi_{\mathbf{l}, \mathbf{i}}$.

Note that $\Psi_{\mathbf{l}, \mathbf{i}} \in H^N(\mathbf{V})$ - the Hilbert space obtained as a completion of the space of N -th degree polynomials in variables $\int \psi(\mathbf{k}) \widehat{\mathbf{V}}(t, \mathbf{x}, \mathbf{k})$ with respect to the standard L^2 norm.

Proposition 1 *For any $(t_1, \mathbf{x}_1), \dots, (t_N, \mathbf{x}_N) \in R \times R^d$ and $p > 0$, $\Psi_{\mathbf{l}, \mathbf{i}}$ belongs to $L^p(\Omega)$ and*

$$(\mathbf{E} |\Psi_{\mathbf{l}, \mathbf{i}}|^p)^{1/p} \leq C \left(\mathbf{E} |\Psi_{\mathbf{l}, \mathbf{i}}|^2 \right)^{1/2} \quad (16)$$

with the constant C depending only on p, N and the dimension d . Moreover, $\Psi_{\mathbf{l}, \mathbf{i}}$ is differentiable in the mean square sense with

$$\nabla \Psi_{\mathbf{l}, \mathbf{i}}(t_1, \dots, t_N, \mathbf{x}_1, \dots, \mathbf{x}_N) = (-1)^{l_j} \int \cdots \int \mathbf{k}_j \psi(\mathbf{k}_1, \dots, \mathbf{k}_N) \quad (17)$$

$$\widehat{V}_{l_1, i_1}(t_1, \mathbf{x}_1, d\mathbf{k}_1) \cdots \widehat{V}_{1-l_j, i_j}(t_j, \mathbf{x}_j, d\mathbf{k}_j) \cdots \widehat{V}_{l_N, i_N}(t_N, \mathbf{x}_N, d\mathbf{k}_N).$$

The proof of Proposition 1 is standard and follows directly from the well known hypercontractivity property for Gaussian measures (see, e.g., [5], Theorem 5.1. and its corollaries), so we do not repeat it here.

The field \mathbf{V} is Markovian i.e.

$$\mathbf{E} \left[\int \psi(\mathbf{k}) \widehat{\mathbf{V}}_l(t, \mathbf{x}, d\mathbf{k}) \mid \mathcal{V}_{-\infty, s} \right] = \int e^{-|\mathbf{k}|^{2\beta}(t-s)} \psi(\mathbf{k}) \widehat{\mathbf{V}}_l(s, \mathbf{x}, d\mathbf{k}), \quad l = 0, 1, \quad (18)$$

for all $\psi \in \mathcal{S}(R^d, R)$, where $\mathcal{V}_{a,b}$ denotes the σ -algebra generated by random variables $\mathbf{V}(t, \mathbf{x})$, for $t \in [a, b]$ and $\mathbf{x} \in R^d$.

To calculate a mathematical expectation of multiple product of Gaussian random variables, it is convenient to use a graphical representation, borrowed from quantum field theory. We refer to, e.g., Glimm and Jaffe [4] and Janson [5]. A *Feynman diagram* \mathcal{F} (of order $n \geq 0$ and rank $r \geq 0$) is a graph consisting of a set $B(\mathcal{F})$ of n vertices and a set $E(\mathcal{F})$ of r edges without common endpoints. So there are r pairs of vertices, each joined by an edge, and $n - 2r$ unpaired vertices, called *free vertices*. $B(\mathcal{F})$ is a set of positive integers. An edge whose endpoints are $m, n \in B$ is represented by \widehat{mn} (unless otherwise specified, we always assume $m < n$); and an edge includes its endpoints. A diagram \mathcal{F} is said to be *based on* $B(\mathcal{F})$. Denote the set of free vertices by $A(\mathcal{F})$, so $A(\mathcal{F}) = \mathcal{F} \setminus E(\mathcal{F})$. The diagram is *complete* if $A(\mathcal{F})$ is empty and *incomplete*, otherwise. Denote by $\mathcal{G}(B)$ the set of all diagrams based on B , by $\mathcal{G}_c(B)$ the set of all complete diagrams based on B and by $\mathcal{G}_i(B)$ the set of all incomplete diagrams based on B . A diagram $\mathcal{F}' \in \mathcal{G}_c(B)$ is called a *completion* of $\mathcal{F} \in \mathcal{G}_i(B)$ if $E(\mathcal{F}) \subseteq E(\mathcal{F}')$.

Let $B = \{1, 2, 3, \dots, n\}$. Denote by $\mathcal{F}_{|k}$ the sub-diagram of \mathcal{F} , based on $\{1, \dots, k\}$. Define $A_k(\mathcal{F}) = A(\mathcal{F}_{|k})$. A special class of diagrams, denoted by $\mathcal{G}_s(B)$, plays an important role in the subsequent analysis: a diagram \mathcal{F} of order n belongs to $\mathcal{G}_s(B)$ if $A_k(\mathcal{F})$ is not empty for all $k = 1, \dots, n$.

We shall adopt the following multiindex notation. For any $P \in Z^+$, multiindex $\mathbf{n} = (n_1, \dots, n_P)$, $|\mathbf{n}|$ stands for $\sum n_p$. If $P' \leq P$ we denote $\mathbf{n}_{|P'} := (n_1, \dots, n_{P'})$. In addition if k is any number we set $\mathbf{n} \cdot k := (n_1, \dots, n_P, k)$.

We work out the conditional expectation for multiple spectral integrals using the Markov property (18).

Proposition 2 *For any function $\psi \in \mathcal{H}$ and $l_1, \dots, l_N \in \{0, 1\}$, $i_1, \dots, i_N \in \{1, \dots, d\}$,*

$$\mathbf{E} \left[\int \dots \int \psi(\mathbf{k}_1, \dots, \mathbf{k}_N) \widehat{V}_{l_1, i_1}(t, \mathbf{x}_1, d\mathbf{k}_1) \dots \widehat{V}_{l_N, i_N}(t, \mathbf{x}_N, d\mathbf{k}_N) \mid \mathcal{V}_{-\infty, s} \right] = \quad (19)$$

$$\sum_{\mathcal{F} \in \mathcal{G}(\{1, \dots, N\})} \int \dots \int \exp \left\{ - \sum_{m \in A(\mathcal{F})} |\mathbf{k}_m|^{2\beta} (t-s) \right\} \psi(\mathbf{k}_1, \dots, \mathbf{k}_N) \widehat{V}_{s, \mathbf{x}_1, \dots, \mathbf{x}_N}(d\mathbf{k}_1, \dots, d\mathbf{k}_N; \mathcal{F})$$

with

$$\begin{aligned} \widehat{V}_{s, \mathbf{x}_1, \dots, \mathbf{x}_N}(d\mathbf{k}_1, \dots, d\mathbf{k}_N; \mathcal{F}) &:= \prod_{m \in A(\mathcal{F})} \widehat{V}_{l_m, i_m}(s, \mathbf{x}_m, d\mathbf{k}_m) \\ &\times \prod_{\widehat{mn} \in E(\mathcal{F})} \left[1 - e^{-(|\mathbf{k}_m|^{2\beta} + |\mathbf{k}_n|^{2\beta})(t-s)} \right] \mathbf{E} [\widehat{V}_{l_m, i_m}(s, \mathbf{x}_m, d\mathbf{k}_m) \widehat{V}_{l_n, i_n}(s, \mathbf{x}_n, d\mathbf{k}_n)]. \end{aligned} \quad (20)$$

Proof. Without loss of generality we consider $\psi(\mathbf{k}_1, \dots, \mathbf{k}_N) = \mathbf{1}_{A_1}(\mathbf{k}_1) \dots \mathbf{1}_{A_N}(\mathbf{k}_N)$ for some Borel sets A_1, \dots, A_N .

Note that $\widehat{\mathbf{V}}_l(t, A_i) = \widehat{\mathbf{V}}_l^0(t, A_i) + \widehat{\mathbf{V}}_l^1(t, A_i)$ where $\widehat{\mathbf{V}}_l^0(t, \cdot)$ is the orthogonal projection of $\widehat{\mathbf{V}}_l(t, \cdot)$ on $L_{-\infty, t}^2$ and $\widehat{\mathbf{V}}_l^1(t, \cdot)$ its complement. Here $L_{a, b}^2$ denotes L^2 closure of the linear span over $\mathbf{V}(s, \mathbf{x})$, $a \leq s \leq b$, $\mathbf{x} \in R^d$. The conditional expectation in (19) equals

$$\sum_{\mathcal{F} \in \mathcal{G}(\{1, \dots, N\})} \prod_{\widehat{mn} \in E(\mathcal{F})} \mathbf{E} [\widehat{V}_{i_m, l_m}^1(t, A_m) \widehat{V}_{i_n, l_n}^1(t, A_n)] \prod_{m \in A(\mathcal{F})} \widehat{V}_{i_m, l_m}^0(t, A_m).$$

The statement follows upon the application of the relations

$$\widehat{\mathbf{V}}_l^0(t, A) = \int_A e^{-|\mathbf{k}|^{2\beta}(t-s)} \widehat{\mathbf{V}}_l(s, d\mathbf{k})$$

and

$$\begin{aligned} \mathbf{E} [\widehat{\mathbf{V}}_l^1(t, A) \otimes \widehat{\mathbf{V}}_{l'}^1(t, B)] &= \\ \int_A \int_B \delta_{l, l'} \left\{ \mathbf{E} [\widehat{\mathbf{V}}_l(t, d\mathbf{k}) \otimes \widehat{\mathbf{V}}_{l'}(t, d\mathbf{k}')] - \mathbf{E} [\widehat{\mathbf{V}}_l^0(t, d\mathbf{k}) \otimes \widehat{\mathbf{V}}_{l'}^0(t, d\mathbf{k}')] \right\}. \end{aligned}$$

■

3 Proof of tightness.

We begin with the following lemma which shows, among other things, that the family of continuous trajectory processes $\mathbf{x}_\varepsilon(t)$, $t \geq 0$ is tight.

Lemma 1 *For the family of trajectories given by (4) we have*

$$\lim_{\varepsilon \downarrow 0} \mathbf{E} [(\mathbf{x}_\varepsilon(t) - \mathbf{x}_\varepsilon(\tau)) \otimes (\mathbf{x}_\varepsilon(t) - \mathbf{x}_\varepsilon(\tau))] = \mathbf{D}|t - \tau|^{2H}$$

where H , \mathbf{D} are given by (8), (9) respectively.

Proof. Thanks to the stationarity of the path $\mathbf{x}_\varepsilon(t)$ it is enough to prove the lemma for $\tau = 0$. By the stationarity of $\mathbf{V}(s, \varepsilon \mathbf{x}(s))$ ([11]), we write

$$\lim_{\varepsilon \downarrow 0} \mathbf{E} [\mathbf{x}_\varepsilon(t) \otimes \mathbf{x}_\varepsilon(t)] = \lim_{\varepsilon \downarrow 0} \varepsilon^2 \int_0^{\frac{t}{\varepsilon^{2\delta}}} ds \int_0^s \mathbf{E} [\mathbf{V}(s', \varepsilon \mathbf{x}(s')) \otimes \mathbf{V}(0, \mathbf{0})] ds' \quad (21)$$

which equals

$$2 \sum_{n=1}^N \mathcal{I}_n + \mathcal{R}_N \quad (22)$$

where

$$\mathcal{I}_n = \varepsilon^{n+1} \int_0^{\frac{t}{\varepsilon^{2\delta}}} ds \int_0^s ds_1 \cdots \int_0^{s_{n-1}} \mathbf{E} [\mathbf{W}_{n-1}(s_1, \dots, s_n, \mathbf{0}) \otimes \mathbf{V}(0, \mathbf{0})] ds_n$$

and

$$\mathbf{W}_0(s_1, \mathbf{x}) = \mathbf{V}(s_1, \mathbf{x})$$

$$\mathbf{W}_n(s_1, \dots, s_{n+1}, \mathbf{x}) = \mathbf{V}(s_{n+1}, \mathbf{x}) \cdot \nabla \mathbf{W}_{n-1}(s_1, \dots, s_n, \mathbf{x}) \quad \text{for } n = 1, 2, \dots$$

with the remainder term

$$\mathcal{R}_N = 2\varepsilon^{N+2} \int_0^{\frac{t}{\varepsilon^{2\delta}}} ds \int_0^s ds_1 \cdots \int_0^{s_N} \mathbf{E} [\mathbf{W}_N(s_1, \dots, s_{N+1}, \varepsilon \mathbf{x}(s_{N+1})) \otimes \mathbf{V}(0, \mathbf{0})] ds_{N+1}. \quad (23)$$

Estimates of \mathcal{I}_n . Elementary calculations show that

$$\lim_{\varepsilon \downarrow 0} \mathcal{I}_1 = \mathbf{D}t^{2H}, \quad \text{for } \alpha + \beta > 1. \quad (24)$$

Since \mathbf{V} is Gaussian we have that

$$\mathbf{E} \mathcal{I}_n = \mathbf{0}, \quad \text{for even } n.$$

We now show that

$$\lim_{\varepsilon \downarrow 0} \mathbf{E} \mathcal{I}_n = \mathbf{0}, \quad \text{for odd } n. \quad (25)$$

Set

$$\mathbf{E}_{s_{n+1}} W_{n-1,i}(s_1, \dots, s_n, \mathbf{x}) := \mathbf{E} [W_{n-1,i}(s_1, \dots, s_n, \mathbf{x}) \mid \mathcal{V}_{-\infty, s_{n+1}}].$$

The i, j -th entry of the matrix \mathcal{I}_n is given by

$$\varepsilon^{n+1} \int_0^{\frac{t}{\varepsilon^{2\delta}}} ds \int_0^s ds_1 \cdots \int_0^{s_{n-1}} \mathbf{E} [\mathbf{E}_0 W_{n-1,i}(s_1, \dots, s_n, \mathbf{0}) V_j(0, \mathbf{0})] ds_n. \quad (26)$$

The conditional expectation in (26) can be expressed in terms of spectral measures of the velocity field. To do so we introduce first the so-called *proper* functions of order $n, \sigma : \{1, \dots, n\} \rightarrow \{0, 1\}$ that appear in the statement of the next lemma. The proper function of order 1 is unique and is given by $\sigma(1) = 0$. Any proper function, σ' , of order $n+1$ is generated from a proper function σ of order n as follows. For some $p \leq n$,

$$\begin{aligned} \sigma'(n+1) &:= 0 \\ \sigma'(k) &:= \sigma(k) \quad \text{for } k \leq n \text{ and } k \neq p \\ \sigma'(p) &:= 1 - \sigma(p). \end{aligned} \quad (27)$$

In other words, each proper function σ of order n generates n different proper functions of order $n+1$. Thus, the total number of proper functions of order n is $(n-1)!$. In the sequel, we sometimes write σ_k instead of $\sigma(k)$.

Lemma 2 *Let $n \geq 1$ and $s_1 \geq s_2 \geq \dots \geq s_n \geq s_{n+1}$, $i \in \{1, \dots, d\}$, $\mathbf{x} \in R^d$. We have then that*

$$\begin{aligned} &\mathbf{E}_{s_{n+1}} W_{n-1,i}(s_1, \dots, s_n, \mathbf{x}) = \\ &\sum \int \cdots \int \varphi_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) \exp\left\{-\sum_{m \in A_n(\mathcal{F})} |\mathbf{k}_m|^{2\beta} (s_n - s_{n+1})\right\} \times \\ &P_{n-1}(\mathcal{F}) Q(\mathcal{F}) \prod_{m \in A_n(\mathcal{F})} \widehat{V}_{i_m, \sigma_m}(s_{n+1}, \mathbf{x}, d\mathbf{k}_m), \end{aligned} \quad (28)$$

where $\varphi_{\mathbf{i},\sigma}^{(n)}$ are some functions with $\sup |\varphi_{\mathbf{i},\sigma}^{(n)}| \leq 1$

$$P_{n-1}(\mathcal{F}) = \prod_{j=1}^{n-1} \left(\sum_{m \in A_j(\mathcal{F})} |\mathbf{k}_m| \right) \exp\left\{-\sum_{m \in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta} (s_j - s_{j+1})\right\} \quad (29)$$

$$Q(\mathcal{F}) = \prod_{\widehat{mm'} \in E(\mathcal{F})} \mathbf{E} \left[\widehat{V}_{i_m, \sigma_m}(0, d\mathbf{k}_m) \widehat{V}_{i_{m'}, \sigma_{m'}}(0, d\mathbf{k}_{m'}) \right]. \quad (30)$$

The summation is over all multiindices \mathbf{i} of length n , whose first component equals i , all $\mathcal{F} \in \mathcal{G}_s$ and all proper functions σ of order n .

Before proving Lemma 2, we apply it to show (25). By (28)

$$\int_0^{\frac{t}{\varepsilon^{2\delta}}} ds \int_0^s ds_1 \cdots \int_0^{s_{n-1}} \mathbf{E}_0 W_{n-1,i}(s_1, \dots, s_n, \mathbf{0}) ds_n =$$

$$\sum \int_0^{\frac{t}{\varepsilon^{2\delta}}} ds \int \cdots \int \tilde{\varphi}_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) \exp\left\{-\sum_{m \in A_n(\mathcal{F})} |\mathbf{k}_m|^{2\beta} s\right\} P'_{n-1}(\mathcal{F}) Q(\mathcal{F}) \prod_{m \in A_n(\mathcal{F})} \hat{V}_{i_m, \sigma_m}(0, d\mathbf{k}_m)$$

for $i = 1, \dots, d$. Here, adopting the convention $s_{n+1} := 0$, we set

$$\tilde{\varphi}_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) := \frac{\int_0^s ds_1 \cdots \int_0^{s_{n-1}} \varphi_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) \times \prod_{j=1}^n \exp\left\{-\sum_{m \in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta} (s_j - s_{j+1})\right\} ds_1 \cdots ds_n}{\int_0^s ds_1 \cdots \int_0^s \prod_{j=1}^n \exp\left\{-\sum_{m \in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta} s_j\right\} ds_1 \cdots ds_n}$$

and

$$P'_{n-1}(\mathcal{F}) = \prod_{j=1}^{n-1} \left\{ \left(\sum_{m \in A_j(\mathcal{F})} |\mathbf{k}_m| \right) \times \frac{1 - \exp\left\{-\sum_{m \in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta} s\right\}}{\sum_{m \in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta}} \right\}.$$

It is elementary to check that, due to $|\varphi_{\mathbf{i},\sigma}^{(n)}| \leq 1$,

$$|\tilde{\varphi}_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n)| \leq 1. \quad (31)$$

By Lemma 2 the left hand side of (26) equals

$$2\varepsilon^{n+1} \sum \int_0^{\frac{t}{\varepsilon^{2\delta}}} ds \int \cdots \int \frac{\tilde{\varphi}_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n)}{\sum_{m \in A_n(\mathcal{F})} |\mathbf{k}_m|^{2\beta}} P'_{n-1}(\mathcal{F}) Q(\mathcal{F}) \mathbf{E} \left[\prod_{m \in A_n(\mathcal{F}) \cup \{n+1\}} \hat{V}_{i_m, \sigma_m}(0, d\mathbf{k}_m) \right]. \quad (32)$$

Here the summation extends over all multiindices $\mathbf{i} = (i_1, \dots, i_{n+1})$ such that $i_1 = i$, $i_{n+1} = j$, all Feynman diagrams $\mathcal{F} \in \mathcal{G}_s$ and all proper functions σ of order n . Note that

$$\frac{1 - e^{-\xi/\varepsilon^{2\delta}}}{\xi} \leq \frac{C}{\varepsilon^{2\delta} + \xi} \quad (33)$$

for all positive ε, ξ . Here and in the sequel C stands for a generic constant independent of ε . C in (33) is also independent of $\xi > 0$. Thus, the absolute value of (32) is bounded by

$$t\varepsilon^{n+1-2\delta} \sum \int_0^K \cdots \int_0^K \frac{p_{n-1,\varepsilon}(\mathcal{F})}{\varepsilon^{2\delta} + \sum_{m \in A_n(\mathcal{F})} k_m^{2\beta}} \prod_{\overline{mm'}} \frac{\delta(k_m - k_{m'}) dk_m dk_{m'}}{k_m^{2\alpha-1}} \quad (34)$$

with

$$p_{n-1,\varepsilon}(\mathcal{F}) := \prod_{j=1}^{n-1} \frac{\sum_{m \in A_j(\mathcal{F})} k_m}{\varepsilon^{2\delta} + \sum_{m \in A_j(\mathcal{F})} k_m^{2\beta}}.$$

Using the fact that

$$\frac{\sum_{m \in A_j(\mathcal{F})} k_m}{\varepsilon^{2\delta} + \sum_{m \in A_j(\mathcal{F})} k_m^{2\beta}} \leq C \frac{\varepsilon^{\frac{\delta}{\beta}} + k_{m_j}}{\varepsilon^{2\delta} + k_{m_j}^{2\beta}}, \quad \forall m_j \in A_j(\mathcal{F}), \quad (35)$$

we have

$$\frac{p_{n-1,\varepsilon}(\mathcal{F})}{\varepsilon^{2\delta} + \sum_{m \in A_n(\mathcal{F})} k_m^{2\beta}} \leq C \prod_{j=1}^{n-1} \frac{\varepsilon^{\frac{\delta}{\beta}} + k_{m_j}}{\varepsilon^{2\delta} + k_{m_j}^{2\beta}} \frac{1}{\varepsilon^{2\delta} + k_{m_n}^{2\beta}}, \quad \forall m_j \in A_j(\mathcal{F}). \quad (36)$$

Bounds (36) and (34) imply that

$$|\mathcal{I}_n| \leq C t \varepsilon^{n+1-2\delta} \sum_{\mathcal{F} \in \mathcal{G}_c} \prod_{\widehat{mm'} \in E(\mathcal{F})} \int_0^K \frac{\left(\varepsilon^{\frac{\delta}{\beta}} + k_m\right)^{q_m}}{\left(\varepsilon^{2\delta} + k_m^{2\beta}\right)^{q_m + \delta_{m,m_n}}} \frac{dk_m}{k_m^{2\alpha-1}} \quad (37)$$

where q_m are certain nonnegative exponents satisfying

$$\sum_{\widehat{mm'} \in E(\mathcal{F})} q_m = n - 1 \quad (38)$$

and $\delta_{m,m_n} = 0$ if $m \neq m_n$ and $= 1$ otherwise.

The integrals appearing in the expression (37) are of the form

$$\int_0^K \frac{\left(\varepsilon^{\frac{\delta}{\beta}} + k\right)^q}{\left(\varepsilon^{2\delta} + k^{2\beta}\right)^{q+r}} \frac{dk}{k^{2\alpha-1}} \quad (39)$$

for some $q \geq 0$ and $r \in \{0, 1\}$. They may diverge or remain bounded as $\varepsilon \downarrow 0$ depending on the exponents q, r . If q, r are such that the integral diverges then $2\beta(q+r) + 2\alpha > 2 + q$ and, consequently,

$$\int_0^{+\infty} \frac{(1+k)^q}{(1+k^{2\beta})^{q+r}} \times \frac{dk}{k^{2\alpha-1}} < +\infty.$$

In either case, the integral (39) is bounded from above by $C\varepsilon^{n(2-\alpha-2\beta)/(\alpha+2\beta-1)}$ for $1 < \alpha+2\beta$ after a change of variable $k'_j = k_j/\varepsilon^{\frac{\delta}{\beta}}$ in case (39) diverges as $\varepsilon \rightarrow 0$. Therefore

$$|\mathcal{I}_n| \leq C t \varepsilon^{n+1-2\delta} \varepsilon^{\frac{n(2-\alpha-2\beta)}{\alpha+2\beta-1}-1} \leq C t \varepsilon^{\frac{n-2}{\alpha+2\beta-1}} \quad (40)$$

which vanish as $\varepsilon \downarrow 0$ for $n \geq 3$.

Estimates of \mathcal{R}_N . By (23)

$$\mathcal{R}_N = 2\varepsilon^{N+2} \int_0^{\frac{t}{\varepsilon^{2\delta}}} ds \int_0^s ds_1 \cdots \int_0^{s_N} \mathbf{E} \left[\mathbf{E}_{s_{N+1}} \mathbf{W}_N(s_1, \dots, s_{N+1}, \varepsilon \mathbf{x}(s_{N+1})) \otimes \mathbf{V}(0, \mathbf{0}) \right] ds_{N+1}.$$

By the Cauchy-Schwartz inequality we get that

$$|\mathcal{R}_N|^2 \leq 4t^2 \varepsilon^{4(1-\delta)+2N} \mathbf{E} |\mathbf{V}(0, \mathbf{0})|^2 \times \quad (41)$$

$$\max_{0 \leq s \leq t/\varepsilon^{2\delta}} \mathbf{E} \left| \int_{s \geq s_1 \geq \dots \geq s_{N+1} \geq 0} \cdots \int \mathbf{E}_{s_{N+1}} \mathbf{W}_N(s_1, \dots, s_N, s_{N+1}, \varepsilon \mathbf{x}(s_{N+1})) ds_1 \cdots ds_{N+1} \right|^2.$$

The stationarity of the Lagrangian velocity field implies that the maximum in (41) is equal to

$$\begin{aligned} & \max_{0 \leq s \leq t/\varepsilon^{2\delta}} \mathbf{E} \left| \int_0^s ds' \int_{s' \geq s_1 \geq \dots \geq s_N \geq 0} \dots \int \mathbf{E}_0 \mathbf{W}_N(s_1, \dots, s_N, 0, \mathbf{0}) ds_1 \dots ds_N \right|^2 \leq \quad (42) \\ & C \max_{0 \leq s \leq t/\varepsilon^{2\delta}} \mathbf{E} \left| \int_0^s ds' \int_{s' \geq s_1 \geq \dots \geq s_N \geq 0} \dots \int \mathbf{E}_0 \nabla \mathbf{W}_{N-1}(s_1, \dots, s_{N-1}, s_N, \mathbf{0}) ds_1 \dots ds_N \right|^2 \times \mathbf{E} |\mathbf{V}(0, \mathbf{0})|^2. \end{aligned}$$

Here the hypercontractive property of the Gaussian measure is used. Subsequent applications of Lemma 2 to (42) yields the upper bound

$$C \frac{t^2}{\varepsilon^{4\delta}} \mathbf{E} \left| \sum_{\mathcal{F} \in \mathcal{G}_s, \sigma, \mathbf{i}} \int \dots \int \psi_{\mathbf{i}, \sigma}(\mathbf{k}_1, \dots, \mathbf{k}_N) P_N(\mathcal{F}) Q(\mathcal{F}) \prod_{m \in A_N(\mathcal{F})} \widehat{V}_{i_m, \sigma_m}(0, d\mathbf{k}_m) \right|^2 \quad (43)$$

with some $|\psi_{\mathbf{i}, \sigma}| \leq 1$. The summation above extends over all Feynman diagrams $\mathcal{F} \in \mathcal{G}_s$, the relevant proper functions σ and multiindices \mathbf{i} .

Thus, we have

$$\mathcal{R}_N^2 \leq C t^4 \varepsilon^{2N+4(1-2\delta)} \sum_{\mathcal{F}, \mathcal{F}'} \int_0^K \dots \int_0^K p_{N, \varepsilon}(\mathcal{F}) p_{N, \varepsilon}(\mathcal{F}') \prod_{\widehat{mm'}} \frac{\delta(k_m - k_{m'}) dk_m dk_{m'}}{k_m^{2\alpha-1}}. \quad (44)$$

Here the summation extends over all possible completions of with $\mathcal{F} \in \mathcal{G}_s(\{1, \dots, N\})$, $\mathcal{F}' \in \mathcal{G}_s(\{N+1, \dots, 2N\})$. The product is over all edges of any completion of $\mathcal{F} \cup \mathcal{F}'$. Arguing as for (37) we obtain that

$$|\mathcal{R}_N|^2 \leq C t^4 \varepsilon^{2N+4(1-2\delta)} \sum_{\mathcal{F}, \mathcal{F}'} \prod_{\widehat{mm'}} \int_0^K \left(\frac{\varepsilon^{\frac{\delta}{\beta}} + k_m}{\varepsilon^{2\delta} + k_m^{2\beta}} \right)^{q_m} \frac{dk_m}{k_m^{2\alpha-1}} \quad (45)$$

for some $q_m \geq 0$ with

$$\sum_{\widehat{mm'}} q_m = 2N. \quad (46)$$

Moreover,

$$|\mathcal{R}_N|^2 \leq C t^4 \varepsilon^{2N+4(1-2\delta)} \varepsilon^{-\frac{2N(2-\alpha-2\beta)}{\alpha+2\beta-1}} \leq C t^4 \varepsilon^{\frac{2N}{\alpha+2\beta-1}+4(1-2\delta)}. \quad (47)$$

which vanishes as $\varepsilon \downarrow 0$ for a sufficiently large N . In conclusion, we proved that the left hand side of (21) tends to $\mathbf{D}t^{2H}$ as $\varepsilon \downarrow 0$, provided that $\alpha + \beta > 1$ (see (24)).

By the hypercontractivity property of the L^p norms over Gaussian measures we also know that for any $p \geq 1$ and $T > 0$ there exists a constant $C > 0$

$$\mathbf{E} |\mathbf{x}_\varepsilon(t) - \mathbf{x}_\varepsilon(s)|^p \leq C(t-s)^{2Hp} \quad (48)$$

for any $T \geq t \geq s \geq 0$, $\varepsilon > 0$.

Proof of Lemma 2. We prove the lemma by induction. The case $n = 1$ is obvious by choosing $\varphi_i^{(0)} \equiv 1$. Suppose that the result holds for n . For the sake of convenience we assume with no loss of any generality that $s_{n+2} = 0$, then

$$\mathbf{E}_0 W_{n+1, i}(s_1, \dots, s_{n+1}, \mathbf{x}) = \mathbf{E}_0 \left\{ \mathbf{V}(s_{n+1}, \mathbf{x}) \cdot \nabla \mathbf{E}_{s_{n+1}} W_{n, i}(s_1, \dots, s_n, \mathbf{x}) \right\}. \quad (49)$$

By virtue of the inductive assumption we can represent $\mathbf{E}_{s_{n+1}} W_{n,i}$ using (28) and as a result (49) becomes

$$\sum \mathbf{E}_0 \left[\int \cdots \int \varphi_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) \exp\left\{-\sum_{m \in A_n(\mathcal{F})} |\mathbf{k}_m|^{2\beta} (s_n - s_{n+1})\right\} P_{n-1}(\mathcal{F}) Q(\mathcal{F}) \right. \\ \left. \widehat{\mathbf{V}}_0(s_{n+1}, \mathbf{x}, d\mathbf{k}_{n+1}) \cdot \nabla \left\{ \prod_{m \in A_n(\mathcal{F})} \widehat{V}_{i_m, \sigma_m}(s_{n+1}, \mathbf{x}, d\mathbf{k}_m) \right\} \right]. \quad (50)$$

To calculate (50) we decompose each $\widehat{V}_{\sigma,i}(s, \mathbf{x}, d\mathbf{k})$ as

$$\widehat{V}_{\sigma,i}(s, \mathbf{x}, d\mathbf{k}) = \widehat{V}_{\sigma,i}^0(s, \mathbf{x}, d\mathbf{k}) + \widehat{V}_{\sigma,i}^1(s, \mathbf{x}, d\mathbf{k}) \quad (51)$$

where

$$\widehat{V}_{\sigma,i}^0(s, \mathbf{x}, d\mathbf{k}) = e^{-|\mathbf{k}|^{2\beta}(s-t)} \widehat{V}_{\sigma,i}(t, \mathbf{x}, d\mathbf{k}) \quad (52)$$

is the orthogonal projection of $\widehat{V}_{\sigma,i}$ on $\mathcal{V}_{-\infty,t}$. Expression (50) becomes

$$\sum \mathbf{E}_0 \left[\int \cdots \int \varphi_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) \exp\left\{-\sum_{m \in A_n(\mathcal{F})} |\mathbf{k}_m|^{2\beta} (s_n - s_{n+1})\right\} P_{n-1}(\mathcal{F}) Q(\mathcal{F}) \mathcal{K}(\mathcal{F}) \right] \quad (53)$$

with

$$\mathcal{K}(\mathcal{F}) := \sum_{\substack{\varrho = \{\varrho_j\} \\ j \in A_n(\mathcal{F}) \cup \{n+1\}}} \widehat{\mathbf{V}}_0^{\varrho_{n+1}}(s, \mathbf{x}, d\mathbf{k}_{n+1}) \cdot \nabla \left\{ \prod_{m \in A_n(\mathcal{F})} \widehat{V}_{\sigma_m, i_m}^{\varrho_m}(s, \mathbf{x}, d\mathbf{k}_m) \right\}.$$

The term corresponding to $\varrho_j \equiv 1$ vanishes, as is clear from the following calculation,

$$\mathbf{E} \left\{ \int \cdots \int \varphi_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) P_{n-1}(\mathcal{F}) Q(\mathcal{F}) \right. \quad (54)$$

$$\left. \widehat{\mathbf{V}}_0^1(s, \mathbf{x}, d\mathbf{k}_{n+1}) \cdot \nabla \left(\prod_{m \in A_n(\mathcal{F})} \widehat{V}_{\sigma_m, i_m}^1(s, \mathbf{x}, d\mathbf{k}_m) \right) \right\} =$$

$$\nabla \cdot \mathbf{E} \left\{ \int \cdots \int \varphi_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) P_{n-1}(\mathcal{F}) Q(\mathcal{F}) \widehat{\mathbf{V}}_0^1(s, \mathbf{x}, d\mathbf{k}_{n+1}) \prod_{m \in A_n(\mathcal{F})} \widehat{V}_{\sigma_m, i_m}^1(s, \mathbf{x}, d\mathbf{k}_m) \right\} = 0$$

by homogeneity of the velocity field. By (12)-(11)

$$\widehat{\mathbf{V}}_0(s, \mathbf{x}, d\mathbf{k}_{n+1}) \cdot \nabla \left\{ \prod_{m \in A_n(\mathcal{F})} \widehat{V}_{\sigma_m, i_m}(s, \mathbf{x}, d\mathbf{k}_m) \right\} \quad (55) \\ = \sum_{m' \in A_n(\mathcal{F})} \mathbf{k}_{m'} \cdot \widehat{\mathbf{V}}_0(s, \mathbf{x}, d\mathbf{k}_{n+1}) \prod_{m \in A_n(\mathcal{F})} \widehat{V}_{\sigma_m^{m'}, i_m}(s, \mathbf{x}, d\mathbf{k}_m)$$

where

$$\sigma_m^{m'} := \begin{cases} 1 - \sigma_{m'} & \text{if } m' = m \\ \sigma_m & \text{otherwise.} \end{cases} \quad (56)$$

By (52), (51), (55) and the definition (29), (53) further reduces to

$$\begin{aligned} & \sum \int \cdots \int \sum_{i_{n+1}=1}^d \sum_{m' \in A_n(\mathcal{F})} \sum_{\mathcal{F}'} \varphi_{\mathbf{i}, \sigma}^{(n)} \frac{k_{m', i_{n+1}}}{\sum_{m \in A_n(\mathcal{F})} |\mathbf{k}_m|} \\ & \exp\left\{-\sum_{m \in A(\mathcal{F}')} |\mathbf{k}_m|^{2\beta} s_{n+1}\right\} P_n(\mathcal{F}) Q(\mathcal{F}) \prod_{m \in A(\mathcal{F}')} \hat{V}_{\sigma_{m'}, i_m}(t, \mathbf{x}, d\mathbf{k}_m) \\ & \prod_{\hat{p}\hat{q} \in E(\mathcal{F}')} \left[1 - e^{-(|\mathbf{k}_p|^{2\beta} + |\mathbf{k}_q|^{2\beta})(s-t)}\right] \mathbf{E} \left[\hat{V}_{\tilde{\sigma}_p, m', i_p}(0, \mathbf{0}, d\mathbf{k}_p) \hat{V}_{\tilde{\sigma}_q, m', i_q}(0, \mathbf{0}, d\mathbf{k}_q)\right] \Big\} \end{aligned} \quad (57)$$

with $\tilde{\sigma}_{1, m'} = 0$ and $\tilde{\sigma}_{j+1, m'} = \sigma_j^{m'}$ and all incomplete Feynman diagrams \mathcal{F}' based on the set $A_n(\mathcal{F}) \cup \{n+1\}$.

Lemma 2 follows with

$$\varphi_{\mathbf{i}, \sigma_{m'}}^{(n+1)}(\mathbf{k}_1, \dots, \mathbf{k}_{n+1}) := \varphi_{\mathbf{i}, \sigma}^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) \frac{k_{m', i_{n+1}}}{\sum_{m' \in A_n(\mathcal{F})} |\mathbf{k}_{m'}|} \prod_{\hat{p}\hat{q} \in E(\mathcal{F}')} \left[1 - e^{-(|\mathbf{k}_p|^{2\beta} + |\mathbf{k}_q|^{2\beta})s_{n+1}}\right]$$

□

4 Proof of weak convergence

It is easy to see that the Gaussian processes

$$\mathbf{y}_\varepsilon(t) := \varepsilon \int_0^{\frac{t}{\varepsilon^{2q}}} \mathbf{V}(s, \mathbf{0}) ds \quad t \geq 0. \quad (58)$$

converge weakly to the fractional Brownian Motion $\mathbf{B}_H(t)$, $t \geq 0$ given by (7). In addition we have

$$\limsup_{\varepsilon \downarrow 0} \mathbf{E} |\mathbf{y}_\varepsilon(t)|^p < +\infty$$

for any $p \geq 1$, $t \geq 0$.

We now prove that

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \mathbf{E} \{ [x_{\varepsilon, i_1}(t_1) - x_{\varepsilon, i_1}(t_2)]^{p_1} \cdots [x_{\varepsilon, i_M}(t_M) - x_{\varepsilon, i_M}(t_{M+1})]^{p_M} \} \\ & = \mathbf{E} \{ [B_{H, i_1}(t_1) - B_{H, i_1}(t_2)]^{p_1} \cdots [B_{H, i_M}(t_M) - B_{H, i_M}(t_{M+1})]^{p_M} \} \end{aligned} \quad (59)$$

which, in conjunction with the tightness, identifies the fractional Brownian motion $B_H(t)$ as the limit. Equation (59) is a consequence of

$$\lim_{\varepsilon \downarrow 0} |\mathbf{E} \{ [x_{\varepsilon, i_1}(t_1) - x_{\varepsilon, i_1}(t_2)]^{p_1} \cdots [x_{\varepsilon, i_M}(t_M) - x_{\varepsilon, i_M}(t_{M+1})]^{p_M} - \quad (60)$$

$$[y_{\varepsilon, i_1}(t_1) - y_{\varepsilon, i_1}(t_2)]^{p_1} \cdots [y_{\varepsilon, i_M}(t_M) - y_{\varepsilon, i_M}(t_{M+1})]^{p_M} \} = 0$$

with $\mathbf{y}_\varepsilon(t) = (y_{\varepsilon, 1}(t), \dots, y_{\varepsilon, d}(t))$, which, in turn, follows from the next lemma.

Lemma 3 For any positive integers M, p_j , multiindices $\mathbf{i}_j \in \{1, \dots, d\}^{p_j}$ with $j = 1, \dots, M$ we have

$$\lim_{\varepsilon \downarrow 0} \left| \mathbf{E} \left[Z_{\varepsilon, \mathbf{i}_1}^{(p_1)}(t_2, t_1) \cdots Z_{\varepsilon, \mathbf{i}_M}^{(p_M)}(0, t_M) - W_{\varepsilon, \mathbf{i}_1}^{(p_1)}(t_2, t_1) \cdots W_{\varepsilon, \mathbf{i}_M}^{(p_M)}(0, t_M) \right] \right| = 0. \quad (61)$$

Here for any integer $N \geq 1$, multiindex $\mathbf{i} = (i_1, \dots, i_N) \in \{1, \dots, d\}^N$ and $t \geq s$ we define

$$Z_{\varepsilon, \mathbf{i}}^{(N)}(s, t) := \varepsilon^N \int \cdots \int_{\Delta_N(s, t)} \prod_{p=1}^N V_{i_p}(s_p, \varepsilon \mathbf{x}(s_p)) ds_1 \cdots ds_N$$

and

$$W_{\varepsilon, \mathbf{i}}^{(N)}(s, t) := \varepsilon^N \int \cdots \int_{\Delta_N(s, t)} \prod_{p=1}^N V_{i_p}(s_p, \mathbf{0}) ds_1 \cdots ds_N,$$

with $\Delta_N(s, t) := \{(s_1, \dots, s_N) : t/\varepsilon^{2\delta} \geq s_1 \geq \cdots \geq s_N \geq s/\varepsilon^{2\delta}\}$.

Proof. To avoid cumbersome expressions that may obscure the essence of the proof we consider only the special case of $M = 1$ and $t_1 = t, t_2 = 0$. The general case follows from exactly the same argument. We shall proceed with the induction argument on $p_1 = P$. The case when $P = 1$ is trivial because the stationarity of the relevant processes implies that the expression under the limit in (61) vanishes. By (48) we know that

$$\limsup_{\varepsilon \downarrow 0} \mathbf{E} |Z_{\varepsilon, \mathbf{i}}^{(1)}(0, t)|^q < +\infty, \quad \forall q \geq 1.$$

Suppose that (61) is true for $P \geq 2$ and that

$$\limsup_{\varepsilon \downarrow 0} \mathbf{E} |Z_{\varepsilon, \mathbf{i}}^{(P-1)}(0, t)|^q < +\infty, \quad \forall q \geq 1. \quad (62)$$

Like (22) we write

$$\mathbf{E} Z_{\varepsilon, \mathbf{i}}^{(P)}(0, t) = \sum_{n=0}^{N-1} \mathcal{I}_n(0, t) + \mathcal{R}_N(0, t) \quad (63)$$

with

$$\mathcal{I}_n(0, t) := \varepsilon^{P+n+1} \int \cdots \int_{\Delta_P^{(n)}(0, t)} \mathbf{E} \left\{ \mathbf{E}_{s_2} W_{i_1}^n(\mathbf{s}_1^{(n)}, \varepsilon \mathbf{x}(s_2)) \prod_{p=2}^P V_{i_p}(s_p, \varepsilon \mathbf{x}(s_p)) \right\} d\mathbf{s}_1^{(n)} ds_2 \cdots ds_P \quad (64)$$

$$\mathcal{R}_N(0, t) := \varepsilon^{P+N+1} \int \cdots \int_{\Delta_P^{(N)}(0, t)} \mathbf{E} \left\{ \mathbf{E}_{s_{1, N+1}} W_{i_1}^N(\mathbf{s}_1^{(N)}, \varepsilon \mathbf{x}(s_{1, N+1})) \prod_{p=2}^P V_{i_p}(s_p, \varepsilon \mathbf{x}(s_p)) \right\} d\mathbf{s}_1^{(N)} ds_2 \cdots ds_P. \quad (65)$$

Here

$$\Delta_P^{(n)}(s, t) := \{(\mathbf{s}_1^{(n)}, s_2, \dots, s_P) : t/\varepsilon^{2\delta} \geq \mathbf{s}_1^{(n)} \geq s_2 \cdots \geq s_P \geq s/\varepsilon^{2\delta}\}$$

with $\mathbf{s}_1^{(n)} := (s_{1,1}, \dots, s_{1,n+1})$. We write $t \geq \mathbf{s}_1 \geq s$, if $t \geq s_1 \geq s_n \geq s$, where $\mathbf{s} = (s_1, \dots, s_n)$ is any ordered n -tuple in the sense that $s_1 \geq \cdots \geq s_n$.

The argument of the proof of Lemma 2 and (62) imply that

$$\lim_{\varepsilon \downarrow 0} \mathcal{I}_n(0, t) = 0, \quad n \geq 1$$

and

$$\lim_{\varepsilon \downarrow 0} \mathcal{R}_N(0, t) = 0$$

for N sufficiently large. Thus $\mathbf{E}Z_{\varepsilon, \mathbf{i}}^{(P)}(0, t)$ has the same limit as the term

$$\mathcal{I}_0(0, t) := \varepsilon^{P+1} \int \cdots \int_{\Delta_P^{(0)}(0, t)} \mathbf{E} \left\{ V_{i_1}(s_1, \varepsilon \mathbf{x}(s_2)) \prod_{p=2}^P V_{i_p}(s_p, \varepsilon \mathbf{x}(s_p)) \right\} ds_1 \cdots ds_P. \quad (66)$$

For (66) we use a generalization of the argument of the proof of Lemma 2. Let us introduce some additional notation. For any multiindex $\mathbf{i} = (i_1, \dots, i_p)$ and $p \geq 1$ we define $W_{\mathbf{i}}^{p, n}$ by induction as follows. We set

$$W_{i_1, \dots, i_p}^{p, 0}(s_1, \dots, s_p, \mathbf{x}) := V_{i_1}(s_1, \mathbf{x}) \cdots V_{i_p}(s_p, \mathbf{x}) - \mathbf{E}\{V_{i_1}(s_1, \mathbf{x}) \cdots V_{i_p}(s_p, \mathbf{x})\}$$

and

$$W_{i_1, \dots, i_p}^{p, n+1}(s_1, \dots, s_{p-1}, \mathbf{s}_p^{(n+1)}, \mathbf{x}) := \nabla W_{i_1, \dots, i_p}^{p, n}(s_1, \dots, \mathbf{s}_p^{(n)}, \mathbf{x}) \cdot \mathbf{V}(s_{p, n+2}, \mathbf{x})$$

for any ordered $(n+1)$ -tuple $\mathbf{s}_p^{(n)} = (s_{p,1}, \dots, s_{p,n+1}) \leq s_{p-1}$ and $(n+2)$ -tuple $\mathbf{s}_p^{(n+1)} = (s_{p,1}, \dots, s_{p,n+1}, s_{p,n+2})$. Expanding the left hand side of (66) like (22) we obtain that

$$\begin{aligned} \mathcal{I}_0(0, t) = & \varepsilon^{P+1} \int \cdots \int_{\Delta_P^{(0)}(0, t)} \mathbf{E} \{ V_{i_1}(s_1, \varepsilon \mathbf{x}(s_2)) V_{i_2}(s_2, \varepsilon \mathbf{x}(s_2)) \} \mathbf{E} \left\{ \prod_{p=3}^P V_{i_p}(s_p, \varepsilon \mathbf{x}(s_p)) \right\} ds_1 ds_2 \cdots ds_P + \\ & \sum_{n=0}^{N-1} \mathcal{I}_{1,n}(0, t) + \mathcal{R}_{1,N}(0, t) \end{aligned}$$

where

$$\mathcal{I}_{1,n}(0, t) := \quad (67)$$

$$\varepsilon^{P+n+1} \int \cdots \int_{\Delta_P^{(1,n)}(0, t)} \mathbf{E} \left\{ \mathbf{E}_{s_3} W_{i_1, i_2}^{2, n}(s_1, \mathbf{s}_2^{(n)}, \varepsilon \mathbf{x}(s_3)) \prod_{p=3}^P V_{i_p}(s_p, \varepsilon \mathbf{x}(s_p)) \right\} ds_1 ds_2^{(n)} ds_2 \cdots ds_P$$

$$\mathcal{R}_{1,N}(0, t) := \quad (68)$$

$$\varepsilon^{P+N+1} \int \cdots \int_{\Delta_P^{(1,N)}(0, t)} \mathbf{E} \left\{ \mathbf{E}_{s_{2,N+1}} W_{i_1, i_2}^{2, N}(s_1, \mathbf{s}^{(N)}, \varepsilon \mathbf{x}(s_{2,N+1})) \prod_{p=3}^P V_{i_p}(s_p, \varepsilon \mathbf{x}(s_p)) \right\} ds_1 ds_2^{(N)} ds_3 \cdots ds_P,$$

$$\Delta_P^{(1,n)}(0, t) := \{(s_1, \mathbf{s}_2^{(n)}, s_3, \dots, s_P) : t/\varepsilon^{2\delta} \geq s_1 \geq \mathbf{s}_2^{(n)} \geq \dots \geq s_P \geq 0\}.$$

We represent the conditional expectations appearing in (67) and (68) using a generalization (Lemma 4) of Lemma 2.

To formulate it we need a generalized notion of a proper function, which we call a p -proper function. Let p be a positive integer. The p -proper function of order 1 is unique and is given by $\sigma(i) = 0, i = 1, \dots, p$. Any p -proper function, σ' , of order $n+1$ is generated from a p -proper function σ of order n as follows. For some $q \leq p+n$,

$$\begin{aligned}\sigma'(p+n+1) &:= 0 \\ \sigma'(k) &:= \sigma(k) \quad \text{for } k \leq n+p \text{ and } k \neq q \\ \sigma'(q) &:= 1 - \sigma(q).\end{aligned}\tag{69}$$

We also distinguish a special class of Feynman diagrams $\mathcal{G}_s^p(B)$: a diagram \mathcal{F} of order $n+p$ belongs to $\mathcal{G}_s^p(B)$ if $A_k(\mathcal{F})$ is not empty for all $k = p, \dots, n+p$.

Lemma 4 *For any positive integer p , $s_1 \geq \dots \geq s_{p-1} \geq \mathbf{s}_p^{(n-1)} \geq s$, a multiindex $\mathbf{i} = (i_1, \dots, i_p) \in \{1, \dots, d\}^p$ we have*

$$\mathbf{E}_s W_{\mathbf{i}}^{p,n-1}(s_1, \dots, s_{p-1}, \mathbf{s}_p^{(n-1)}, \mathbf{x}) = \sum \int \dots \int \varphi_{\mathbf{j},\sigma}^{(p,n)}(\mathbf{k}_1, \dots, \mathbf{k}_{p+n}) \tag{70}$$

$$\exp\left\{-\sum_{m \in A_{n+p}(\mathcal{F})} |\mathbf{k}_m|^{2\beta}(s_{p,n} - s)\right\} P_{p,n-1}(\mathcal{F}) Q(\mathcal{F}) \prod_{m \in A_{n+p}(\mathcal{F})} \widehat{V}_{i_m, \sigma_m}(s, \mathbf{x}, d\mathbf{k}_m),$$

where $\varphi_{\mathbf{j},\sigma}^{(p,n)}$ are functions satisfying $|\varphi_{\mathbf{j},\sigma}^{(p,n)}| \leq 1$ and

$$\begin{aligned}P_{p,n}(\mathcal{F}) &= \prod_{j=p}^{n+p-1} \left(\sum_{m \in A_j(\mathcal{F})} |\mathbf{k}_m| \right) \exp\left\{-\sum_{m \in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta}(s_{p,j-p} - s_{p,j-p+1})\right\}, \\ Q(\mathcal{F}) &= \prod_{\widehat{mm'} \in E(\mathcal{F})} \mathbf{E} \left[\widehat{V}_{i_m, \sigma_m}(0, d\mathbf{k}_m) \widehat{V}_{i_{m'}, \sigma_{m'}}(0, d\mathbf{k}_{m'}) \right].\end{aligned}\tag{71}$$

The summation is over all multiindices $\mathbf{j} = (j_1, \dots, j_{n+p})$, such that $\mathbf{j}|_p = \mathbf{i}$, all $\mathcal{F} \in \mathcal{G}_s^p$ and all p -proper functions σ of order n . Here by a convention $s_{p,0} := s_{p-1}$.

The proof of Lemma 4 is exactly the same as that of Lemma 2 and is omitted.

Continue the proof of Lemma 3 using Lemma 4 we have that $\mathcal{I}_0(0, t)$ is asymptotically equal to $\mathbf{E} Z_{\varepsilon, \mathbf{i}}^{(P)}(0, t)$ and

$$\varepsilon^{P+1} \int \dots \int_{\Delta_P(0, t)} \mathbf{E} \left\{ V_{i_1}(s_1, \varepsilon \mathbf{x}(s_1)) V_{i_2}(s_2, \varepsilon \mathbf{x}(s_2)) \prod_{p=3}^P V_{i_p}(s_p, \varepsilon \mathbf{x}(s_p)) \right\} ds_1 ds_2 \dots ds_P$$

is asymptotically equal to $\mathbf{E} Z_{\varepsilon, \mathbf{i}}^{(P)}(0, t)$, as $\varepsilon \downarrow 0$. Repeating the above argument p -times we obtain (61). Finally the hypercontractivity properties of the L^p norms over Gaussian measure space imply that (62) holds with $P-1$ replaced by P_{\square}

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